CALCULATION OF THE PRINCIPAL PARAMETERS OF FREE SUPERSONIC JETS OF AN IDEAL COMPRESSIBLE FLUID

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An approximate method of determining the principal parameters of axially symmetrical underexpanded jets is set out. The results are presented in the form of computing formulas for the geometrical characteristics of the configuration of the shock waves and the boundaries of the jet. On the basis of experience with these calculations, some simple approximating relationships are recommended.

Notation: x,y, rectangular coordinates in the plane of the axial section; p, pressure; ρ, density; w, modulus of the velocity; M, Mach number; k, polytropy index; n, degree of wastage of the jet; α, Mach angle; ϑ, angle between the velocity vector and the symmetry axis; ϑ_a, aperture semiangle of the nozzle in the outlet section; θ, angle of rotation of the flow in the shock wave; φ, inclination of the leading edge of the shock wave to the symmetry axis; ω, angle between the incident jump and the velocity vector of the incident flow; K, curvature of the current lines; R, radius of curvature.

The indices denote: a, parameters in the outlet section of the nozzle; °, parameters at the boundary of the jet; *, parameters in front of the shock-wave branch point; 1, parameters at the vertex of the angle between the leading edges of the branched jumps; 2, parameters behind the central jump. The dimensionless gas-dynamic quantities are referred to the corresponding retardation parameters at the outlet from the nozzle, the quantities with dimensions of velocity to the maximum velocity of steady-state outflow into a vacuum. The linear quantities are expressed in terms of the radius of the outlet cross section of the nozzle.

1. Numerical calculations of the shape of the boundaries in supersonic jets of a nonviscous, nonheat-conducting gas are laborious and cumbersome. Simple approximate methods of solving this problem are therefore very much to be sought. One such method was proposed in [1]; it was based on expanding the solution in series with respect to even powers of the angle ϑ. The expansion is achieved by transforming to an auxiliary plane of independent variables (pressure-current function) [2]. The initial approximation then gives a unidimensional solution for flows in a channel with a variable cross-sectional area. In the next approximation, the isobars (including the isobaric current line) form a family of second-order curves. However, since the flow closely resembles a plane flow around an obstacle in the neighborhood of the sharp edge [3] (up to distances of the order of the radius of the nozzle from the center of expansion), the generator of the jet boundary differs little from a straight line up to distances of the same order. We may therefore employ a twofold analytical specification for the shape of the boundary generator in different regions

$$y=\begin{cases}1+(\operatorname{tg}\vartheta^{\circ})\,x, & x\leqslant x^{\circ}\\ \sqrt{A+Bx+Cx^{2}}, & x\geqslant x^{\circ}\end{cases}\tag{1.1}$$

where A, B, C, x^0, ϑ^0 are parameters reflecting the influence of the specific conditions of outflow and the internal structure of the jet. The parameters A, B, C, and x° were defined in [1] in terms of the coordinates x_*, y_* of the branch point of the shock waves and the pressure p_1 at the tip of the angle between the incident and reflected jumps. The following conditions and assumptions also apply:

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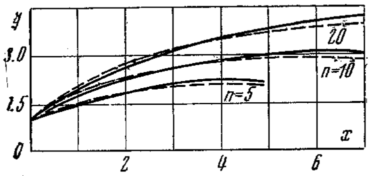

Fig. 1

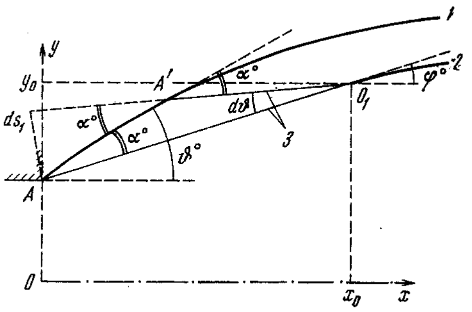

Fig. 2

a) smooth joining of the sections of the jet boundary generator is assumed at the point (x°, y°)

$$y^\circ = 1 + x^\circ \operatorname{tg}\vartheta^\circ = \sqrt{A + Bx^\circ + Cx^{\circ 2}}$$
$$\frac{dy}{dx} = \operatorname{tg}\vartheta^\circ = \frac{B + 2Cx^\circ}{2y^\circ}$$

b) the active cross section of the flow passing through the contour of the central jump is regarded as approximately plane and normal to the symmetry axis; since this section lies fairly close to the maximal cross section, the direction of the velocity vector in the former differs little from axial, and the angle ϑ is therefore regarded as small within this section;

c) allowing for the foregoing approximation, an integral form is taken for the law of conservation governing the mass of the gas;

d) it is considered that the curvature of the current lines K and the gradients of the pressure p remain finite within the region of flow considered and constitute reasonably smooth functions of distance from the boundary of the jet;

e) in the calculations, these functions are approximated by a quadratic relationship for p and a linear relationship for K.

In this way we obtain an equation in x° and formulas for the coefficients A, B, and C

$$y_m^2 - 2(x_* - x^\circ)(1 + x^\circ \operatorname{tg}\vartheta^\circ)\operatorname{tg}\vartheta^\circ - (x^\circ \operatorname{tg}\vartheta^\circ + 1)^2 - (x_* - x^\circ)^2\left[\operatorname{tg}^2\vartheta^\circ + \frac{2y_m^3(p_1/p^\circ - 1)}{kM^{\circ 2}(1 + x^\circ \operatorname{tg}\vartheta^\circ)^2(y_m - y_*)}\right] = 0 \quad (1.2)$$
$$A = y_m^2 - Bx_* - Cx_*^2 = (1 + x^\circ \operatorname{tg}\vartheta^\circ)^2 - Bx^\circ - Cx^{\circ 2}$$
$$B = 2\left[(1 + x^\circ \operatorname{tg}\vartheta^\circ)\operatorname{tg}\vartheta^\circ - Cx^\circ\right]$$
$$C = (x_* - x^\circ)^{-2}\left[y_m^2 - 2(x_* - x^\circ)(1 + x^\circ \operatorname{tg}\vartheta^\circ)\operatorname{tg}\vartheta^\circ - (1 + x^\circ \operatorname{tg}\vartheta)^2\right]$$

where

$$y_m = -\frac{y_*}{3k}\left[\frac{1}{k}\left(\frac{p_1}{p^\circ} - 1\right) + 2\right]^{-1}\left\{\left(\frac{p_1}{p^\circ} - 1\right) - \frac{kM^{\circ 2}}{y_*}\left[\frac{y_*^2}{k^2M^{\circ 4}}\left(\frac{p_1}{p^\circ} - 1\right)^2 + \frac{3}{M^{\circ 2}}\left[\frac{1}{k}\left(\frac{p_1}{p^\circ} - 1\right) + 2\right]\left\{\frac{y_*^2}{M^{\circ 2}}\left[\frac{5}{k}\left(\frac{p_1}{p^\circ} - 1\right) + 6\right] + \frac{6D}{p^{\circ\gamma}}\left(\frac{2}{M^{\circ 2}} + k - 1\right)\right\}\right]^{1/2}\right\}$$
$$D = \frac{1}{2}\left\{p_a^\gamma\left(\sqrt{1 - p_a^{1-\gamma}} - 1\right) + \frac{k+1}{2k}p_a - \left[p_*^\gamma\left(\sqrt{1 - p_*^{1-\gamma}} - 1\right) + \frac{k+1}{2k}p_*\right]y_*^2 + \frac{k-1}{2k}(1 - y_*^2)p^\circ\right\} \qquad \left(\gamma = \frac{1}{k}\right)$$

Figure 1 illustrates various comparisons between the calculation of the boundary generator based on Eqs. (1.1) and (1.2) (broken lines) and the calculation based on the method of characteristics [3] (continuous line). In this we took $k = 1.4$, $M_a = 1.5$, $\vartheta_a = 0$.

Analogous results were obtained for other values of the original parameters ($1 \leq M_a \leq 3.5$, $5 \leq n \leq 25$).

2. In an underexpanded jet, the first shock wave, or the suspended jump (curve 2 in Fig. 2), is generated at a certain distance from the nozzle cutoff, at the point O_1, at which the characteristics of the second family 3 reflected from the boundary of the jet 1 first intersect. The coordinates of this point may easily be expressed in terms of the radius of curvature R° of the initial element of the jet boundary on the assumption that the characteristics are practically linear up to the intersection point.

A simple approximate formula may be obtained for R° from the equation of motion projected on the normal to the current line, allowing for the continuity equation

$$\frac{\rho w^2}{R} = -\frac{\partial p}{\partial n}, \qquad \frac{\partial \vartheta}{\partial n} + \frac{1}{\rho w}\frac{\partial \rho w}{\partial s} = -\frac{\sin\vartheta}{y}$$

Here R is the local radius of curvature, and s and n are the distances along the current line and the normal to the latter. Since ρw = const along the boundary, from the foregoing equations we obtain

$$\frac{\rho w^2}{R} = \frac{\sin\vartheta}{y}\frac{\partial p}{\partial \vartheta}$$

The derivative $dp/d\vartheta$ is calculated along the normal to the boundary. The change in pressure in the elementary compression wave reflected from the boundary may be approximately calculated from the plane theory of small perturbations. The flow in the neighborhood of the boundary is vortex-free; hence,

$$dp \approx \frac{\rho w^2}{\sqrt{M^2-1}} d\vartheta$$

Thus,

$$R^\circ = \frac{\sqrt{M^2-1}}{\sin\vartheta^\circ} = \frac{\operatorname{ctg}\alpha^\circ}{\sin\vartheta^\circ}$$

Here α° is the Mach angle, and ϑ° is the initial inclination of the boundary generator to the axis of the jet. The approximate formula (2.1) gives entirely satisfactory agreement with the results of exact calculations [3]. According to Eq. (2.1) and Fig. 2, the coordinates of the point of generation of the suspended jump are given by

$$x_0 = \frac{\cos\alpha^\circ}{\sin\vartheta^\circ}\cos(\vartheta^\circ - \alpha^\circ), \qquad y_0 = 1 + \frac{\cos\alpha^\circ}{\sin\vartheta^\circ}\sin(\vartheta^\circ - \alpha^\circ)$$

In the neighborhood of the triple point, the parameters are very sensitive to any change in the initial inclination of the suspended jump to the axis of the jet φ°. The arc of the suspended jump is of a considerable extent; hence even small errors in determining the angle φ° may give rise to serious errors at the end of this arc. Let us calculate the increment in the characteristic angle $(\vartheta - \alpha)$ on moving along the latter characteristic, descending from the sharp edge of the nozzle. In the neighborhood of this edge, the derivatives of the gas-dynamic quantities measured along the characteristics of the first family in the region of the wave of rarefaction are, in absolute magnitude, far greater than the same quantities in the direction of the characteristics of the second family. Hence the compatibility relationship along the characteristics of the first family is approximately satisfied in all directions, in the same form as in the case of plane flows

$$d\vartheta + \frac{\cos^2\alpha}{{}^1/_2(k-1) + \sin^2\alpha} d\alpha = 0 \tag{2.2}$$

This relationship holds, in particular, along the initial element of the boundary characteristic, on which the exact relationship for the characteristics of the second family is satisfied

$$d\vartheta - \frac{\cos^2\alpha}{{}^1/_2(k-1) + \sin^2\alpha} d\alpha - \sin\alpha\sin\vartheta\, dl = 0 \tag{2.3}$$

Here dl is an element of displacement along the characteristic in question. It follows from (2.2) and (2.3) that

$$\frac{d(\vartheta - \alpha)}{dl} = \frac{k+1}{4}\frac{\sin\alpha\sin\vartheta}{\cos^2\alpha}$$

The latter formula enables us to introduce a linear correction to the angle

$$\varphi^\circ = \vartheta^\circ - \alpha^\circ + \left[\frac{d(\vartheta-\alpha)}{dl}\right]^\circ l^\circ = \vartheta^\circ - \alpha^\circ + \frac{k+1}{4}\operatorname{tg}\alpha^\circ$$

Calculations showed that due allowance for this correction constituted one of the main requirements in increasing the accuracy of determination of the parameters governing the configuration of the shock waves in the jet.

3. If, for the sake of clarity, the boundary of the jet is likened to a solid wall, then the suspended jump may be treated as a shock wave arising when a supersonic flow passes around a concave surface. The greater the Mach number in the "incident" flow, the more closely does this wave approach the surface around

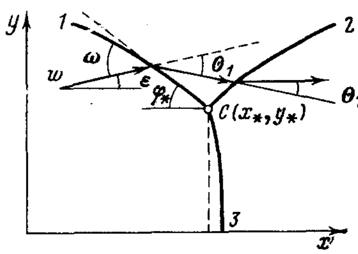

Fig. 3

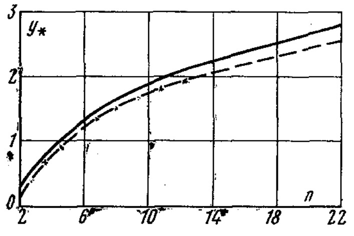

Fig. 4

which the flow is occurring. On this basis, certain authors have used the limiting hypersonic approximation, assuming that the suspended jump coincides with the boundary of the jet. For large but finite Mach numbers in front of the suspended jump, the generator of the latter differs little in shape from the line forming the boundary, and may be geometrically derived from this by means of a relatively slight deformation. This deformation may be approximately carried out analytically by changing several parameters in the structural relationship defining the shape of the jet boundary.

Let us suppose that the arc of the boundary generator is given by the equation of a three-parameter curve of form

$$f(x,\ y,\ a,\ b,\ c) = 0 \tag{3.1}$$

where a, b, c are parameters determined from the conditions characterizing the jet boundary. Then, according to the foregoing arguments, the equation of the generator giving the surface of the leading edge of the suspended jump takes the same form (3.1), while the parameters a, b, c may be found with due allowance for the properties of this jump, i.e., the curve in question should pass through two points with coordinates (x_0, y_0) and (x_*, y_*); the inclinations of the jump to the symmetry axis at these points, respectively, equal φ° and φ_*. Hence,

$$\begin{gathered} f(x_0,\ y_0,\ a,\ b,\ c) = 0,\ f(x_*,\ y_*,\ a,\ b,\ c) = 0 \\ f_x(x_0,\ y_0,\ a,\ b,\ c) - f_y(x_0,\ y_0,\ a,\ b,\ c) \operatorname{tg} \varphi^\circ = 0 \\ f_x(x_*,\ y_*\ a,\ b,\ c) + f_y(x_*,\ y_*\ a,\ b,\ c) \operatorname{tg} \varphi_* = 0 \end{gathered} \tag{3.2}$$

In accordance with (1.1) we may, in particular, consider that

$$f(x,\ y,\ a,\ b,\ c) = y - \sqrt{a + bx + cx^2} = 0$$

From Eqs. (3.2) we then find

$$\begin{gathered} a = y_0^2 - bx_0 - cx_0^2 = y_*^2 - bx_* - cx_*^2 \\ b = (x_* - x_0)^{-1}[y_*^2 - y^2_0 + (x_0 + x_*)\ (y_0 \operatorname{tg}\varphi^\circ + y_* \operatorname{tg} \varphi_*)] \\ c = -(x_* - x_0)^{-1}(y_0 \operatorname{tg} \varphi^\circ + y_* \operatorname{tg} \varphi_*) \\ \operatorname{tg} \varphi_* = [y_*(x_* - x_0)]^{-1}[(x_* - x_0)y_0 \operatorname{tg} \varphi^\circ - y_*^2 + y_0^2] \end{gathered} \tag{3.3}$$

Equations (3.3) contain the unknown parameters x_* and y_*. We shall now discuss the problem of calculating these.

4. In a jet flow there is always a tendency for the flow to even out in the direction of the jet axis. Even behind the first system of jumps, the inclinations of the velocity vectors to the axis of the jet become very slight. This property also applies to the contact discontinuity descending from the contour of the central jump. To a first approximation this jump may be considered as the first jump of compression. Then the initial inclination of the contact discontinuity equals zero, while the flow may be regarded as unidimensional in front of the leading edge of the jump.

As a second approximation we set up the following scheme (Fig. 3). The velocity vector w in front of the triple point C makes a small angle ε with the symmetry axis; the angle of the incident jump 1 with respect to this vector changes by a small amount $\Delta\omega$; the central jump 3 curves, while the initial element of the contact discontinuity remains parallel to the axis. Hence, in the second approximation, $\varepsilon - \theta_1 + \theta_2 = 0$, where θ_1 and θ_2 are the angles of rotation of the flow in the incident and reflected jumps 1 and 2, respectively.

Let us make use of the formula for the angle of rotation of the flux in the shock wave

$$\operatorname{tg}\theta = \frac{\sigma - 1}{1 + kM^2 - \sigma}\left[\frac{2k}{k+1}\,\frac{M^2}{\sigma + (k-1)/(k+1)} - 1\right]^{1/2} = P(M,\sigma), \quad \sigma = \frac{p_1}{p_*} \tag{4.1}$$

Here p_1 is the pressure at the tip of the angle between the leading edges of the branched jumps 1 and 2, p_* is the pressure in front of the branch point. Let θ_0 and σ_0 be the values of θ and σ in the first approximation. We introduce the function γ in the following form:

$$\gamma = \left[\cos^2\theta_0\left(\frac{d\,\mathrm{tg}\,\theta_0}{d\sigma_0} - \frac{d\,\mathrm{tg}\,\theta_2}{d\sigma_0}\right)\right]^{-1} \tag{4.2}$$

$$\mathrm{tg}\,\theta_0 = P(M, \sigma_0), \quad \mathrm{tg}\,\theta_2 = P(M_1, \sigma_1), \quad \sigma_1(M, \sigma_0) = p_2/p_1 \tag{4.3}$$

Here $M_1(M, \sigma_0)$ is the Mach number at the tip of the angle between the incident and reflected jumps, p_2 is the pressure behind the forward jump. Equations (4.2) and (4.3) determine γ as a function of M. Using this function, we may write the equations for the linear increments of the parameters in the second approximation in a compact form

$$\Delta\sigma = \gamma(M)\,\varepsilon, \quad \Delta\omega = \frac{(k+1)\gamma(M)\,\varepsilon}{2kM^2\sin 2\omega}, \quad \Delta\varphi = \Delta\omega - \varepsilon \tag{4.4}$$

In the current tube passing through the contour of the central jump, we consider the flow in front of the latter as being one-dimensional. Hence,

$$\varepsilon \approx \frac{dy}{dx} = \frac{d\ln y}{dM}\frac{dM}{dx}\,y = \frac{\beta(M)}{\eta(M)}\,y \tag{4.5}$$

where

$$\eta(M) = 1 - \frac{(k+1)\gamma(M)}{2kM^2\sin 2\omega}, \quad \beta(M) = \frac{\eta(M)}{2x'(M)}\frac{d\ln q(M)}{dM}$$

Here q(M) is the tabulated gas-dynamic flow function, x(M) is a function characterizing the distribution of Mach numbers along the axis of the jet. On the basis of (4.4) and (4.5)

$$\varphi_* = \omega + \Delta\varphi = \omega - \beta y_*$$

Substituting the value of φ_* in the last of equations (3.3), we find the radius of the central compression jump

$$y_* = \frac{1}{2}\left(1 - \beta\frac{x_* - x_0}{\cos^2\omega}\right)^{-1}\left\{\left(\left[(x_* - x_0)^2\,\mathrm{tg}^2\,\omega + 4y_0\left(1 - \beta\frac{x_* - x_0}{\cos^2\omega}\right) \times (y_0 + (x_* - x_0)\,\mathrm{tg}\,\varphi^\circ)\right]^{1/2} - (x_* - x_0)\,\mathrm{tg}\,\omega\right\} \tag{4.6}$$

The determination of the distance x_* to the central jump is no serious problem, since there are some quite simple approximations and empirical formulas determining this quantity to a fair accuracy. In calculating the jet parameters by the method proposed, we made particular use of the empirical equation of [3]:

$$x_* = 0.8\{3.04n^{0.437} + 3.1\,[(2M_a^2 - 1)^{1/2} - 1] - 1.1\,(M_a^2 - 1) + 0.65\,[(n-2)\sqrt{M_a^2 - 1}]^{1/2} - 1\} \quad (n \geqslant 2)$$

We made a number of calculations relating to the configuration of shock waves for underexpanded jets over wide ranges of the original parameters. The results of the calculations were compared with published data relating to numerical calculations and experimental measurements. Figure 4 illustrates a comparison between the calculation based on Eq. (4.6) (broken curve) and experimental data [3] (continuous curve), taking

$$k = 1.4, \quad M_a = 1.5, \quad \vartheta_a = 0$$

Analogous results were obtained for other values of the original parameters

$$(1 \leqslant M_a \leqslant 5, \quad 2 \leqslant n \leqslant 100).$$

Our estimation of the influence of the various possible assumptions as to the radius of the central jump showed that the replacement of this by a straight jump ($\beta = 0$) introduced an error of the order of 20-30% into the value of y_*. The displacement of the point of origin of the suspended jump to the edge of the nozzle changed the value of y_* by 30-40%. However, the greatest error was associated with the determination of the initial angle of the suspended jump; an increment of φ° to the angle $\vartheta^\circ - \alpha^\circ$ changed y_* by 50-70%. These errors were capable of occurring all in the same direction rather than cancelling each other.

5. In the foregoing method of determining the principal parameters of underexpanded jets, the computing formulas contain functions $x(M)$, $\sigma(M)$, and $\gamma(M)$; it is relatively difficult to calculate the values of these from the exact relationships. Experience shows that these functions may nevertheless be calculated to a fair accuracy by simple approximate equations.

In the approximate methods of calculating the principal parameters of underexpanded jets, the function $x(M)$ is usually regarded as specified in the form of some analytical relationship. The simpler the form of this relationship, the less troublesome is the calculation of the principal jet parameters. In the range $0 \le x \le 20$ satisfactory agreement with numerical calculations may be obtained with the following approximate formula:

$$M = M_a - 1 + \frac{a\left(x - \sqrt{M_a^2 - 1}\right) + b}{x - \sqrt{M_a^2 - 1} + c}, \quad \begin{aligned} a &= 3.56 + 0.01e^{4.77k} \\ b &= 1 + 0.024e^{3.9k} \\ c &= 1.7 + 0.019e^{4.3k} \end{aligned} \tag{5.1}$$

In order to calculate the ratio of the pressures σ in the configuration of shock waves with one direct jump, we recommend the following approximating equation (accuracy of the order of 1% in the range $2 \le M_a \le 10$)

$$\sigma = \left(\frac{2k}{k+1}M^2 - \frac{k-1}{k-1}\right)\left[a + 0.19\,(b - 0.65c) + \frac{b}{M} - \frac{c}{M^2}\right] \tag{5.2}$$

where

$$\begin{aligned} a &= -0.25 + 0.97\,(k-1) - 1.08\,(k-1)^2 \\ b &= 1.74 - 4.86\,(k-1) + 8.24\,(k-1)^2 \\ c &= 1.19 - 5.85\,(k-1) + 12.16\,(k-1)^2 \end{aligned}$$

The function $\gamma(M)$ introduced earlier may be approximated by a simple formula (accuracy of the order of 5% in the range $2 \le M_a \le 15$)

$$\gamma = -2.99k + 2.5 + (3.17k - 2.78)\,M + (0.08k + 0.16)M^2 \tag{5.3}$$

The use of Eqs. (5.1), (5.2), and (5.3) reduces the calculation of the principal jet parameters to a series of quite elementary calculations.

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